ON IYENGAR-TYPE INEQUALITIES VIA QUASI-CONVEXITY AND QUASI-CONCAVITY

M. EMIN ÖZDEMIR★

Abstract. In this paper, we obtain some new estimations of Iyengar-type inequality in which quasi-convex(quasi-concave) functions are involved. These estimations are improvements of some recently obtained estimations. Some error estimations for the trapezoidal formula are given. Applications for special means are also provided.

1. Short Historical Background and Introduction

If it is necessary to bound one quantity by another, the classical inequalities are very useful for this purpose. This first book called " Inequalities " written by Hardy, Littlewood and Polya at cambridge University Press in 1934 represents the first effort to systemize a rapidly expanding domain. In this sense, the second important book " Classical and New Inequalities in Analysis " is written by D.S. Mitrinović, J.E.Pećarič and A.M. Fink. The third book " Analytic Inequalities " written by D.S. Mitrinović, and the other book " Means and Their Inequalities " written by Bullen, D.S. Mitrinović, D.S. Vasic, P.M.

Today inequalities play a significant role for the development in all fields of Mathematics. They have applications in a variety of applied Mathematics. For example, convex functions are tractable in optimization because local optimality guarantees global optimality. In recent years a number of authors have discovered new integral inequalities for convex, $s-$convex functions, logarithmic convex functions, $h-$convex functions, *quasi*-convex functions, $m-$convex functions, $(\alpha, m)-$convex functions, co-ordinated convex functions, and Godunova-Levin function, $P-$function.

On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883,p.82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality :

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of a real numbers and $a, b \in I$ with $a < b$. If the function f is concave, the inequality in (1.1) is reversed. That is

$$f\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a}\int_a^b f(x)\,dx \geq \frac{f(a)+f(b)}{2}.$$

2000 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Weighted Hölder Inequality, Hölder Inequality, Power-mean Inequality, Differentiable Function, quasi-convex Function.

★Corresponding Author.

For recent results, generalizations and new inequalities related to the inequality (1.1) see ([7]-[17]).

Then left hand side of Hermite-Hadamard inequality (LHH) can also be estimated by the inequality of Iyengar.

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{M(b-a)}{4}-\frac{[f(b)-f(a)]^2}{4M(b-a)} \tag{1.2}$$

where

$$M=\sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right|;x\neq y\right\}$$

In [3], Daniel Alexandru Ion proved the following inequalities of Iyengar type for differentiable $quasi-$convex functions:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right) \tag{1.3}$$

where $f:[a,b]\rightarrow\mathbb{R}$ is differentiable function on (a,b), and $|f'|$ is $quasi-$convex on $[a,b]$ with $a<b$.

and

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ &\frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} \end{aligned} \tag{1.4}$$

where $f:[a,b]\rightarrow\mathbb{R}$ is differentiable function on (a,b), and $|f'|^{\frac{p}{p-1}}$ is $quasi-$convex on $[a,b]$ with $a<b$.

We give some necessary definitions and mathematical preliminaries for $quasi-$convex functions which are used throughout this paper.

Definition 1. (see [1]) A function $f:[a,b]\rightarrow\mathbb{R}$ is said to be $quasi-$convex on $[a,b]$ if

$$f(\lambda x+(1-\lambda)y)\leq\max\{f(x),f(y)\}, \tag{1.5}$$

holds for all $x,y\in[a,b]$ and $\lambda\in[0,1]$.For additional results on $quasi-$convexity, see [2]. Clearly, any convex function is $quasi-$convex function. Furthermore, there exists $quasi-$convex functions which are not convex. See [3] :

$$g(t)=\begin{cases} 1, & t\in[-2,-1] \\ t^2, & t\in(-1,2] \end{cases}$$

is not a convex function on $[-2,2]$,but it is a $quasi-$convex function on $[-2,2]$.If we choose $g:[-2,2]\rightarrow\mathbb{R}$, $g(-2)=1$, $g(2)=4$ and for $\alpha=\frac{1}{2}$, $a=-2$, $b=0$, we get $g(\alpha a+(1-\alpha)b)=g(-1)=1$ and $\alpha g(a)+(1-\alpha)g(b)=\frac{1}{2}g(-2)+\frac{1}{2}g(0)=\frac{1}{2}$.Thus it is not convex but it is $quasi-$convex function for all $\alpha\in[0,1]$, $g(-\alpha 2+(1-\alpha)2)\leq\max\{g(-2),g(2)\}=\max\{1,4\}=4$.

The main purpose of this paper is to point out new estimations of the inequality in (1.2) , but now for the class of $quasi-$convex functions.

In order to prove our main results we need the following lemma (see [4]).

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a, b \in I$ with $a < b$ and f'' be integrable on $[a, b]$. Then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)\, f''(ta+(1-t)b)\, dt.$$

The main results of this paper are given by the following theorems.

2. The Results

Theorem 1. *Let $f : I^\circ \subset [0,\infty) \to \mathbb{R}$, be a twice differentiable mapping on I°, such that $f'' \in L[a,b]$, $a, b \in I$ with $a < b$. If $|f''|^q$ is quasi$-$convex on $[a,b]$ for $q > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.1}$$
$$\leq \frac{(b-a)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}} \left(\beta(p+1, q+1)\right)^{\frac{1}{q}}$$
$$\times \left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta(\ ,\ )$ is Euler Beta Function:

$$\beta(x\ ,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \qquad x, y > 0.$$

Theorem 2. *Let $f : I^\circ \subset [0,\infty) \to \mathbb{R}$, be a twice differentiable mapping on I°, such that $f'' \in L[a,b]$, $a, b \in I$ with $a < b$. If $|f''|^q$ is quasi$-$convex on $[a,b]$ for $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.2}$$
$$\leq \frac{(b-a)^2}{4}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{q-1}{q}} \left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 3. *With the assumptions of Theorem 1, we obtain another*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{2^{1+\frac{1}{q}}}\left(\beta(2, p+1)\right)^{\frac{1}{p}} \left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

3. proof of main results

Proof of Theorem 1: Using Lemma 1 and the well known Hölder's inequality for $q > 1$,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right|$$
$$\leq \frac{(b-a)^2}{2}\left(\int_0^1 t^{\frac{q-p}{q-1}} dt\right)^{\frac{q-1}{q}} \left[\int_0^1 t^p (1-t)^q \left|f''(ta+(1-t)b)\right|^q dt\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

On the other hand, since $|f''|^q$ is $quasi-$convex on $[a,b]$, we know that for any $t\in[0,1]$

$$\left|f''\left(ta+(1-t)\,b\right)\right|^q \leq \max\left\{|f''(a)|^q, |f''(b)|^q\right\}.$$

Therefore, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(b-a)^2}{2}\left(\int_0^1 t^{\frac{q-p}{q-1}}dt\right)^{\frac{q-1}{q}}\left[\int_0^1 t^p(1-t)^q\left|f''\left(ta+(1-t)\,b\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^2}{2}\left(\int_0^1 t^{\frac{q-p}{q-1}}dt\right)^{\frac{q-1}{q}}\left[\int_0^1 t^p(1-t)^q\left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)dt\right]^{\frac{1}{q}} \\
=\ & \frac{(b-a)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta\left(p+1,q+1\right)\right)^{\frac{1}{q}}\left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof.

Corollary 1. *In Theorem 1, if we choose* $M=Sup_{x\in(a,b)}|f''(x)|<\infty$, *we get*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(b-a)^2}{2}M\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta\left(p+1,q+1\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof of Theorem 2: From Lemma 1 and the well known power-mean inequality we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(b-a)^2}{2}\int_0^1 t\,(1-t)\left|f''\left(ta+(1-t)\,b\right)\right|dt \\
\leq\ & \frac{(b-a)^2}{2}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,(1-t)^q\left|f''\left(ta+(1-t)\,b\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^2}{2}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,(1-t)^q\left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)dt\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^2}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(q+1)(q+2)}\right)^{\frac{1}{q}}\left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^2}{4}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}}\left(\max\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof of Theorem 2 is completed.

Corollary 2. *Under the assumptions of Theorem 2,*

Case i: Since $\lim_{q\to\infty}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}}=1$ and $\lim_{q\to1^+}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}}=\frac{1}{3}$, we have

$$\frac{1}{3}<\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}}<1,\qquad q\in[1,\infty).$$

Therefore,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)^2}{4}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}. \tag{3.1}$$

In (3.1),

- if $|f''|^q$ is decreasing, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)^2}{4}|f''(a)|,$$

- if $|f''|^q$ is increasing , we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)^2}{4}|f''(b)|.$$

Case ii: If we choose $M=Sup_{x\in(a,b)}|f''(x)|<\infty$ in (2.2), then the inequality in (2.1) is better than the inequality in (2.2).

Proof of Theorem 3: From Lemma 1 with properties of modulus we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \le\ &\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt.\end{aligned} \tag{3.2}$$

Now, if we use the following weighted version of Hölder's inequality [5, p. 117]:

$$\left|\int_I f(s)g(s)h(s)ds\right|\le\left(\int_I|f(s)|^p h(s)ds\right)^{\frac{1}{p}}\left(\int_I|g(s)|^q h(s)ds\right)^{\frac{1}{q}} \tag{3.3}$$

for $p>1, p^{-1}+q^{-1}=1$, h is nonnegative on I and provided all the other integrals exist and are finite.

If we rewrite the inequality (3.2) with respect to (3.3) with $|f''|^q$ is *quasi*−convex on $[a,b]$ for all $t\in[0,1]$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt \\
=\ & \frac{(b-a)^2}{2}\int_0^1 (1-t)\left|f''(ta+(1-t)b)\right|tdt \\
\leq\ & \frac{(b-a)^2}{2}\left(\int_0^1 (1-t)^p\,tdt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(ta+(1-t)b)\right|^q tdt\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^2}{2}\left(\beta(2,p+1)\right)^{\frac{1}{p}}\left(\frac{\max\left\{|f''(a)|^q,|f''(b)|^q\right\}}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof of Theorem 3 is completed.

Corollary 3. *In Theorem 3, if we choose $M=Sup_{x\in(a,b)}|f''(x)|<\infty$, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^2}{2^{1+\frac{1}{q}}}M\left(\beta(2,p+1)\right)^{\frac{1}{p}}.$$

Remark 1. *From Theorems 1-3, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\min\{v_1,v_2,v_3\}$$

where

$$v_1=\frac{(b-a)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta(p+1,q+1)\right)^{\frac{1}{q}}\times\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}},$$

$$v_2=\frac{(b-a)^2}{4}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{q-1}{q}}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}$$

and

$$v_3=\frac{(b-a)^2}{2^{1+\frac{1}{q}}}\left(\beta(2,p+1)\right)^{\frac{1}{p}}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

4. Error Estimates for the Trapezoidal Rule

Let d be a partition $a=x_0<x_1<x_2<...<x_n=b$ of the interval $[a,b]$ and consider the quadrature formula

$$\int_a^b f(x)dx=T_i(f,d)+E_i(f,d),\qquad i=1,2,...,n-1 \tag{4.1}$$

where

$$T_1(f,d)=\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)$$

for the Trapezoidal version and

$$T_2(f,d)=\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$$

for the Midpoint formula and $E_i(f,d)$ denotes the associated approximation errors.

Proposition 1. *Suppose that all the assumptions of Theorem 1 are satisfied for every division d of $[a,b]$, we have*

$$\begin{aligned} |E(f,d)| &\le \frac{1}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta\left(p+1,q+1\right)\right)^{\frac{1}{q}} \\ &\quad\times\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^3\left(\max\left\{\left|f''(x_i)\right|^q,\left|f''(x_{i+1})\right|^q\right\}\right)^{\frac{1}{q}}. \end{aligned}$$

Proof. Appliying Theorem 1 on the subinterval $(x_{i+1},x_i)\,, i=1,2,...,n-1$ of the partition and by using the $quasi-$convexity of $|f''|^q$, we obtain

$$\begin{aligned} &\left|\frac{f(x_i)+f(x_{i+1})}{2}-\frac{1}{(x_{i+1}-x_i)}\int_{x_i}^{x_{i+1}} f(x)dx\right| \\ &\le \frac{\left(x_{i+1}-x_i\right)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta\left(p+1,q+1\right)\right)^{\frac{1}{q}} \\ &\quad\times\left(\max\left\{\left|f''(x_i)\right|^q,\left|f''(x_{i+1})\right|^q\right\}\right)^{\frac{1}{q}}. \end{aligned}$$

Hence in (4.1), we have

$$\begin{aligned} \left|\int_a^b f(x)dx-T(f,d)\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx-\frac{f(x_i)+f(x_{i+1})}{2}\left(x_{i+1}-x_i\right)\right\}\right| \\ &\le \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)dx-\frac{f(x_i)+f(x_{i+1})}{2}\left(x_{i+1}-x_i\right)\right| \\ &\le \frac{1}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta\left(p+1,q+1\right)\right)^{\frac{1}{q}} \\ &\quad\times\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^3\left(\max\left\{\left|f''(x_i)\right|^q,\left|f''(x_{i+1})\right|^q\right\}\right)^{\frac{1}{q}}. \end{aligned}$$

□

Proposition 2. *Suppose that all the assumptions of Theorem 2 are satisfied for every division d of $[a,b]$, we have*

$$\begin{aligned} |E(f,d)| &\le \frac{1}{4}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}} \\ &\quad\times\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^3\left(\max\left\{\left|f''(x_i)\right|^q,\left|f''(x_{i+1})\right|^q\right\}\right)^{\frac{1}{q}}. \end{aligned}$$

Proof. The proof is immediate follows from Theorem 2 and by applying a similar argument to the Proposition 1. □

Proposition 3. *Suppose that all the assumptions of Theorem 3 are satisfied for every division d of $[a,b]$, we have*

$$\begin{aligned}|E(f,d)| &\leq \frac{1}{2}\left(\beta\left(1,p+1\right)\right)^{\frac{1}{p}}\\ &\quad\times\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^2\left(\frac{\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. The proof is immediate follows from Theorem 3 and by applying a similar argument to the Proposition 1. □

5. Applications to special means

Let us consider the special means for real numbers a,b $(a\neq b)$. We take

1. Arithmetic mean:

$$A(a,b)=\frac{a+b}{2},\quad a,b\in\mathbb{R}.$$

2. Logarithmic mean:

$$L(a,b)=\frac{a-b}{\ln|a|-\ln|b|},\quad |a|\neq|b|,a,b\neq 0,a,b\in\mathbb{R}.$$

3. Generalized log-mean:

$$L_n(a,b)=\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}},\quad n\in\mathbb{Z}\backslash\left\{-1,0\right\},a,b\in\mathbb{R},a\neq b.$$

Proposition 4. *Let $a,b\in\mathbb{R}$, $a<b$ and $n\in\mathbb{N}$, $n\geq 2$. Then we have*

$$\begin{aligned}&\left|A(a^n,b^n)-L_n^n(a,b)\right|\leq\frac{n(n-1)\left(b-a\right)^2}{2}\\ &\times\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta\left(p+1,q+1\right)\right)^{\frac{1}{q}}\left(\max\left\{|a|^{(n-2)q},|b|^{(n-2)q}\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. The assertion follows from Theorem 1 applied to the *quasi*−convex mapping $f(x)=x^n$, $x\in\mathbb{R}$. □

Proposition 5. *Let $a,b\in\mathbb{R}$, $a<b$ and $n\in\mathbb{N}$, $n\geq 2$. Then we have*

$$\begin{aligned}&\left|A(a^n,b^n)-L_n^n(a,b)\right|\leq\frac{n(n-1)\left(b-a\right)^2}{4}\\ &\times\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}}\left(\max\left\{|a|^{(n-2)q},|b|^{(n-2)q}\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. The assertion follows from Theorem 2 applied to the *quasi*−convex mapping $f(x)=x^n$, $x\in\mathbb{R}$. □

Proposition 6. *Let $a,b\in\mathbb{R}$, $a<b$ and $n\in\mathbb{N}$, $n\geq 2$. Then we have*

$$\begin{aligned}&\left|A(a^n,b^n)-L_n^n(a,b)\right|\leq\frac{n(n-1)\left(b-a\right)^2}{2^{1+\frac{1}{q}}}\\ &\times\left(\beta\left(2,q+1\right)\right)^{\frac{1}{p}}\left(\max\left\{|a|^{(n-2)q},|b|^{(n-2)q}\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. The assertion follows from Theorem 3 applied to the *quasi*−convex mapping $f(x)=x^n$, $x\in\mathbb{R}$. □

References

[1] J. Pećarič, F.Proschan, and Y.L.Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press(1992), Inc.

[2] J. Ponstein, Seven kinds of convexity, SIAM Review 9, 115-119 (1967)

[3] D. A. Ion, Some estimates on the Hermite- Hadamard inequality through Quasi- convex functions, Annals of University of Craiova Math. Comp.Sc. Ser A (2007), 82-87.

[4] M. Alomari, M: Darus and S.S.Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute value are $quasi-$convex, RGMIA Res. Rep.Coll.12(2009) Supplement, Article 17, Online: htpp://www.staff.vu.edu.au/RGMIA/v12(E).asp.

[5] S.S.Dragomir, R.P.Agarwal, N.S. Barnett, Inequalities for Beta and Gamma Functions via some classical and new integral inequalities, Journal of Inequalities and Applications, Vol 5, pp.103-165,2000.

[6] D.S. Mitrinović, J.E.Pećarič. A.M. Fink, Classical and new Inequalities in analysis, Kluwer Academic Publishers, 1993, p.106,10,15.

[7] U.S. Kirmaci, M. K. Bakula, M. E. Özdemir and J. Pećarič, Hadamard type inequalities for $s-$convex functions, Applied Mathematics and Computation, 193 (2007), p.106, 26-35.

[8] M.K. Bakula, M.E. Özdemir and J. Pećarič, Hadamard type inequalities for $m-$convex and $(\alpha, m)-$convex functions, Journal of Inequalities in Pure and Applied Mathematics, Vol. 9 (2008), Issue 4, article 96.

[9] E. Set, M.E. Özdemir and S.S.Dragomir, On the Hermite-Hadamard inequality and other integral inequalities involving two functions, Journal of Inequalities and Applications, 2010, Article ID 148102,9 pages,doi.10.1155/2010/148102.

[10] M.E. Özdemir, Ç. Yildiz, A. O. Akdemir, On some new Hadamard type inequalities for Coordinated $quasi-$convex functions, Submitted.

[11] E. Set, M.E. Özdemir, S.S.Dragomir, On Hadamard type inequalities involving several kinds of convexity, Journal of Inequalities and Applications, 2010, Article ID 1286845,12 pages,doi.10.1155/2010/286845.

[12] M. Avci, H. Kavurmaci, M. E. Ozdemir, New inequalities of Hermite-Hadamard type via $s-$convex functions in the second sense with applications, Appl. Math. Comput., 217 (2011) 5171-5176.

[13] M. E. Ozdemir, M. Avci and H. Kavurmaci, Hermite-Hadamard type inequalities via $(\alpha, m)-$convexity, Comput. Math. Appl., 61 (2011) 2614–2620.

[14] M. E. Ozdemir, M. Avci and E. Set, On some inequalities of Hermite-Hadamard type via $m-$convexity, Appl. Math. Lett., 23 (2010) 1065 1070.

[15] H. Kavurmaci, M. Avci and M. E. Ozdemir, New inequalities of Hermite-Hadamard type for convex functions with applications, Journal of Inequalities and Applications 2011, 2011:86.

[16] M.E. Özdemir, A. O. Akdemir, On some Hadamard type inequalities for convex functions on a rectangular box, Journal of non linear Analysis and Application, Volume 2011, Year 2011 Article ID jnaa-00101, 10pages, doi: 10.5899/2011/jnaa-00101, Research Article.

[17] A. O. Akdemir, M.E. Özdemir and S. Varošanec, On some inequalities for $h-$concave functions, Mathematical and Computer Modelling 55 (2012) 746-753.

★ATATÜRK UNIVERSITY, K.K. EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS, 25240, CAMPUS, ERZURUM, TURKEY

E-mail address: emos@atauni.edu.tr